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THE THEORY OF LINEAR DEPENDENCE.

By Maxime Bôcher.

The subject treated in the following pages is not only one which, owing to its numerous applications, is of considerable importance, but, while distinctly elementary, illustrates well one of the most striking tendencies of modern algebraic and analytic work, namely, the tendency not to be satisfied with results which are merely true "in general," *i. e.* with more or less numerous exceptions, but to strive for theorems which are *always* true.* For these reasons, and because, so far as I know, no adequate treatment of the subject as a whole exists, I venture to offer the following presentation to the readers of the ANNALS.

I wish to call special attention to the application made in §2 to the theory of linear equations. The more usual course is to proceed in the reverse order and to deduce the theory of linear dependence from the theory of linear equations. I think the order here adopted has the advantage of separating from one another with more distinctness the various steps in the reasoning.

1. Definitions and General Theorems. Geometrical Illustrations. Linear dependence may be regarded as a generalization of the conception of proportionality. For this purpose we will lay down the following definition of proportionality, which is at once seen to be equivalent to the ordinary definition:

The two sets of constants

$$x_1', \; x_2', \; \ldots \; x_n',$$
$$x_1'', \; x_2'', \; \ldots \; x_n'',$$

are said to be proportional to each other if two constants c_1 *and* c_2, *not both zero, exist such that:*

$$c_1 x_i' + c_2 x_i'' = 0 \qquad (i = 1, 2, \ldots n).$$

Instead of two sets of constants we now consider m sets, and give the following

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*The great importance of this tendency will be apparent if we remember that when we apply a theorem, it is usually to a special case. If we know merely that the theorem is true "in general," we must first consider whether the special case to which we wish to apply it, is not one of the exceptional cases where the theorem fails.

DEFINITION. *The m sets of n constants each:*

$$x_1^{[i]}, x_2^{[i]}, \; . \; . \; . \; . \; x_n^{[i]} \qquad (i = 1, 2, \; . \; . \; . \; m)$$

are said to be linearly dependent if m constants $c_1, c_2, \; . \; . \; . \; c_m$, *not all zero, exist such that:*

$$c_1 x_j' + c_2 x_j'' + \; . \; . \; . \; . + c_m x_j^{[m]} = 0 \qquad (j = 1, 2, \; . \; . \; . \; n).$$

If this is not the case, the sets of quantities are said to be linearly independent.

In the same way we generalize the familiar conception of the proportionality of two functions as follows:

DEFINITION. *The m functions (of any number of independent variables)* $f_1, f_2, \; . \; . \; . \; . \; f_m$ *are said to be linearly dependent if m constants* $c_1, c_2, \; . \; . \; . \; c_m$, *not all zero, exist such that:*

$$c_1 f_1 + c_2 f_2 + \cdot \cdot \cdot \cdot + c_m f_m \equiv 0.^*$$

If this is not the case the functions are said to be linearly independent.†

The following theorems about linear dependence, while almost self evident, are of sufficient importance to deserve explicit statement.

I. *If m sets of constants (or if m functions) are linearly dependent, it is always possible to express one—but not necessarily any one—of them linearly in terms of the others.*

This is seen at once if we remember that at least one of the c's is not zero. The relations (or relation) in which the c's occur can then be divided through by this c.

II. *If there exist among the sets of constants (or among the functions) a smaller number of sets (or of functions) which are linearly dependent, then the m sets (or the m functions) are linearly dependent.*

III. *If any one of the m sets of constants consists exclusively of zeros (or if any one of the m functions is identically zero), the sets (or the functions) are linearly dependent.*

In order to show at once that this conception of linear dependence is important, we will give here a few simple geometrical applications, leaving the reader to supply the proofs of the statements made.

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* We use the symbol ≡ here and in what follows (except in §5 (5) where, as is obvious from the context, it denotes a congruence) to denote an identity, *i. e.* an equality which holds either for all values of the independent variables, or at least for all the values with which we have to deal.

† We might clearly go farther and consider the linear dependence of m sets of n functions each. The two cases of the text would be merely special cases from this general point of view.

The sets of n constants with which we had to deal in the first definition may advantageously be regarded as the homogeneous coordinates of points in space of $n - 1$ dimensions. It will then be convenient to speak of the linear dependence or independence of these points. The geometrical meaning of linear dependence here, will be at once evident from the following theorems for the case $n = 4$:

Two points are linearly dependent when, and only when, they coincide.

Three points are linearly dependent when, and only when, they are collinear.

Four points are linearly dependent when, and only when, they are complanar.

Five or more points are always linearly dependent.

Another geometrical application is suggested by the following considerations:

A set of n ordinary* quantities is nothing more or less than a complex quantity with n principal units. Our first definition of linear dependence is therefore precisely equivalent to the following:

The m complex quantities

$$a_1, a_2, \dots \dots a_m$$

are said to be linearly dependent if m ordinary quantities $c_1, c_2, \dots \dots c_m$ not all zero, exist such that:

$$c_1 a_1 + c_2 a_2 + \dots \dots + c_m a_m = 0.$$

Now the simplest geometrical interpretation for a complex quantity with n principal units is as a vector in space of n dimensions,† and we are thus led to the conception of the linear dependence of vectors. The geometrical meaning of this linear dependence will be seen from the following theorems for the case $n = 3$.

Two vectors are linearly dependent when, and only when, they are collinear.

Three vectors are linearly dependent when, and only when, they are complanar.

Four or more vectors are always linearly dependent.

In order to get a geometrical interpretation of the linear dependence of functions, we must consider, not the functions themselves, but the equations obtained by equating these functions to zero. We speak of these equations as being linearly dependent if the functions are linearly dependent. If then we regard the independent variables as rectangular coordinates these equations give us geometrical loci in space of as many dimensions as there are independent variables. Thus in the cases of two and three variables, we have plane curves and surfaces respectively. The case of two loci is of no interest, as they must coincide in order to be linearly dependent. The following theorems will serve to illustrate the geometrical meaning of linear dependence.

(1.) In the plane:

Three circles are linearly dependent when, and only when, they belong to the same coaxial family.

Four circles are linearly dependent when, and only when, they have a (real or imaginary) common orthogonal circle.

Four circles are linearly dependent when, and only when, the points of intersection, of the first and second, and the points of intersection of the third and fourth, lie on a circle.

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* Two different standpoints are here possible according as we understand the term, *ordinary quantity* to mean, *real quantity*, or *ordinary complex quantity*.

† There are of course other possible geometrical interpretations. Thus in the case $n = 4$ we may regard our complex quantities as quaternions, and consider the meaning of linear dependence of two, three, or four quaternions.

Five or more circles are always linearly dependent.

(2.) In space:

Three planes are linearly dependent when, and only when, they intersect in a line,
Four planes are linearly dependent when, and only when, they intersect in a point.
Five or more planes are always linearly dependent.

2. The Condition for Linear Dependence of Sets of Constants. In considering the m sets of n constants each

$$x_1^{(i)}, x_2^{(i)}, \ldots . x_n^{(i)} \qquad (i = 1, 2, \ldots m).$$

it will be convenient to distinguish between the two cases $m \leqq n$ and $m > n$.

(*a*) $m \leqq n$. We wish here to prove the following fundamental theorem:

The necessary and sufficient condition for linear dependence is that all the m-rowed determinants of the matrix

$$\left\| \begin{matrix} x'_1 & x'_2 & \ldots\ldots & x'_n \\ x''_1 & x''_2 & \ldots\ldots & x''_n \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot \\ x_1^{(m)} & x_2^{(m)} & \ldots\ldots & x_n^{(m)} \end{matrix} \right\|$$

should vanish.

That this is a necessary condition is at once obvious, for if the m sets of constants are linearly dependent, one of the rows can be expressed as a linear combination of the others. Accordingly if in any of the m-rowed determinants we subtract from the elements of this row the corresponding elements of the other rows, each row multiplied by a suitable constant, the elements of this row will reduce to zero. The determinant therefore vanishes.

We come now to the proof that the vanishing of these determinants is also a sufficient condition. We assume, therefore, that all the m-rowed determinants of the above matrix vanish. Let us also assume that at least one of the k-rowed determinants of the matrix does not vanish but that all determinants of order higher than k vanish.* Without any real loss of generality we may (and will) assume that *the k-rowed determinant which stands in the upper left hand corner of the matrix does not vanish;* for by changing the order of the sets of constants and the order of the constants in each set (and these orders are clearly quite immaterial) we can bring one of the non-vanishing k-rowed determinants into this position.

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* In general we shall therefore have $k = m - 1$, but k may have any value less than m. The only case which we here exclude is that in which all the elements of the matrix are zero, a case in which the linear dependence is at once obvious.

We will now prove that the first $k+1$ sets of constants are linearly dependent. From this the linear dependence of the m sets follows by theorem II of the last section.

Let us denote by $c_1, c_2, \ldots . c_{k+1}$ the first minors of the $(k+1)$-rowed determinant which stands in the upper left hand corner of the matrix, and which correspond to the elements of its last column. If now we remember that all the $(k+1)$-rowed determinants vanish we get the relations:

$$c_1 x_j' + c_2 x_j'' + \ldots + c_{k+1} x_j^{[k+1]} = 0 \qquad (j = 1, 2, \ldots n).$$

This formula establishes the linear dependence of the first $k+1$ sets of constants. For $c_{k+1} \neq 0$, being the k-rowed determinant which stands in the upper left hand corner of the matrix.

(b) $m > n$. This case can be reduced to the one already considered by the following simple device. Add to each set of n constants $m-n$ zeros. We then have m sets of m constants each. Their matrix contains only one m-rowed determinant, which vanishes since one, at least, of its columns is composed of zeros. Therefore these m sets of m constants each are linearly dependent; and therefore the original m sets of n constants each were linearly dependent. Thus we get the theorem:

m sets of n constants each are always linearly dependent if $m > n$.

3. Linear Dependence of Polynomials. Linear Homogeneous Equations. Suppose we have m polynomials

$$f_1, f_2, \ldots . f_m$$

in any number of independent variables. The necessary and sufficient condition for the linear dependence of these polynomials is obviously the linear dependence of their m sets of coefficients.* Thus the conditions deduced in the last section can be applied at once to the case of polynomials.

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* It may be noticed that the number of independent variables involved in the polynomials is quite immaterial. What is important is the number of terms. Thus whether we have to deal with the three planes:

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d\ &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0; \end{aligned}$$

or with the three circles:

$$\begin{aligned} a_1(x^2 + y^2) + b_1x + c_1y + d_1 &= 0, \\ a_2(x^2 + y^2) + b_2x + c_2y + d_2 &= 0, \\ a_3(x^2 + y^2) + b_3x + c_3y + d_3 &= 0, \end{aligned}$$

In the theory of simultaneous homogeneous linear equations we have a simple and extremely important application of the results so far obtained. We presuppose here a knowledge of the method of solving by determinants a set of n non-homogeneous linear equations in n unknowns when the determinant of the coefficients of the unknowns is not zero.*

We consider the system of equations:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = 0. \end{array}$$

The values $x_1 = x_2 = \ldots\ldots = x_n = 0$ satisfy these equations in all cases. We shall however in future discard this solution as being trivial, and shall understand by a solution of the equations a set of values of the x's satisfying the equations and not all zero.

The questions we have to answer are first whether the equations have a solution (in the sense just explained); and second how all the solutions which exist can be found. We will take up in succession the cases $m < n$, $m \geqq n$.

(*a*) $m < n$. Here we have the theorem:

The number of equations being less than the number of unknowns the equations always have solutions.

This is at once obvious if we consider the n sets of m constants each:

$$a_{1i},\ a_{2i},\ \ldots\ldots a_{mi} \qquad (i = 1, 2, \ldots\ldots n).$$

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the condition for linear dependence is the vanishing of all the three rowed determinants of the matrix

$$\left\| \begin{array}{cccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right\|.$$

* This method will be found in every book where determinants are touched upon. The treatment, however, in most cases, is incomplete, all that is really proved being that *if the equations have a solution* it is given by the formulæ in question. To complete the treatment it is necessary to prove that the formulæ in question really satisfy the equations, as can readily be done by direct substitution.

These sets of constants are necessarily linearly dependent since $m < n$; *i. e.* there exist n constants $c_1, c_2, \ldots . c_n$, not all zero such that:

$$c_1 a_{j1} + c_2 a_{j2} + \ldots . + c_n a_{jn} = 0 \qquad (j = 1, 2, \ldots . m).$$

i. e. the c's satisfy the above system of linear equations.

(*b*) $m \geqq n$

The number of equations being equal to or greater than the number of unknowns, the necessary and sufficient condition for the existence of solutions is the vanishing of all the n-rowed determinants in the matrix of the coefficients of the equations.

The proof of this theorem is precisely like that of the preceding one.

We come now to the question of actually finding the solutions in the cases in which they exist, and we will begin with the case in which we have fewer equations than unknowns ($m < n$) and in which the equations are linearly independent. There will then be at least one m-rowed determinant in the matrix of coefficients which does not vanish, and we will assume for the sake of clearness that this is the determinant consisting of the first m-columns of the matrix. Let us now write the equations in the form:

$$\begin{array}{l}
a_{11} x_1 + \cdots + a_{1m} x_m = - a_{1\,m+1} x_{m+1} - \cdots - a_{1n} x_n, \\
a_{21} x_1 + \cdots + a_{2m} x_m = - a_{2\,m+1} x_{m+1} - \cdots - a_{2n} x_n, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1} x_1 + \cdots + a_{mm} x_m = - a_{m\,m+1} x_{m+1} - \cdots - a_{mn} x_n.
\end{array}$$

Assigning to $x_{m+1}, x_{m+2}, \ldots x_n$ any values whatever, we can compute from these equations the values of $x_1, x_2, \ldots x_m$; for we have then m non-homogenous equations in m unknowns in which the determinant of the coefficients of the unknowns does not vanish. We thus get the theorem:

If $m < n$ and the equations are linearly independent, to $n - m$ of the unknowns may be assigned arbitrary values and the others will then be completely determined.

It should be noticed that this theorem does not say that *any* $n - m$ of the unknowns can be chosen at pleasure, but merely that a certain set of $n - m$ unknowns can be so chosen,—in fact it is obvious that any set of $n - m$ unknowns may be so chosen which is such that the determinant formed from the coefficients of the remaining m unknowns is not zero.

In all the other cases in which the equations have solutions the equations are linearly dependent, and, even if there are more equations than there are unknowns, all the n-rowed determinants of the matrix of the coefficients vanish. We will now assume that there is at least one k-rowed determinant (which for the sake of clearness we assume as lying in the upper left hand corner of the matrix) which does not vanish, but that all determinants of the matrix of higher order vanish. Denoting then by $f_1, f_2, \ldots f_m$ the linear polynomials which stand on the left hand side of the equations, we see that the $k+1$ functions:

$$f_1, f_2, \ldots f_k, f_l,$$

(where l is any one of the integers $k+1$, $k+2$, ... m) are linearly dependent:

$$c_1 f_1 + c_2 f_2 + \cdots + c_k f_k + c_l f_l \equiv 0.$$

The constant c_l here is not zero as otherwise $f_1, f_2, \ldots f_k$ would be linearly dependent. We may therefore, by dividing through by c_l, express f_l linearly in terms of $f_1, f_2, \ldots f_k$. Accordingly any set of x's which satisfy the first k equations also satisfy the remaining equations. The first k equations however form a system of the sort already discussed, and we get the following theorem which we state in such a form as to include the last theorem stated as a special case:

If the system of equations admits a solution, and if in the matrix of the coefficients at least one k-rowed determinant does not vanish while all determinants of higher order (if there are any) are zero, then $n-k$ of the x's can be chosen at pleasure and the others are then completely determined.

Here again any $n-k$ x's can be chosen at pleasure which have the property that if their coefficients are struck out of the matrix at least one of the k-rowed determinants of the remaining matrix does not vanish.

The only case which is not covered by this theorem is that in which all the coefficients of all the equations are zero, in which case *all* the x's can of course be chosen at pleasure.

In the special case in which $m = n-1$ and the equations are linearly independent we readily get the following theorem:

The solutions of $n-1$ linearly independent homogeneous linear equations in n unknowns are proportional to the $(n-1)$-rowed determinants of the matrix of the coefficients obtained by striking out first the first column, then the second, etc., and taken alternately with the plus and minus sign.

We have here considered only homogeneous equations. Many facts concerning non-homogeneous linear equations can be deduced at once from the theorems here given by letting $x_n = 1$. For a complete treatment of the theory of non-homogeneous linear equations we refer to E. Pascal's admirable little book on determinants §56 (Hoepli, Milan; also a German translation by Leitzmann, Teubner, Leipzig).

4. The Condition for Linear Dependence of Functions of One Variable. Let us consider the n functions

$$f_1(x),\ f_2(x),\ \ldots\ldots f_n(x).$$

We suppose these functions to be defined either throughout a certain region of the complex x-plane, or throughout a certain interval of the real x-axis. In the first case we assume that the functions are analytic throughout the region in question, and therefore have derivatives of all orders at every point of the region; in the second case we assume that, at every point of the interval, each of the functions has a finite derivative of each of the first $n-1$ orders.

If the functions are linearly dependent we have not merely the relation, where the c's are constants not all zero:

$$c_1 f_1 + c_2 f_2 + \cdots\cdots + c_n f_n \equiv 0,$$

but also by differentiation:

$$\begin{array}{l}
c_1 f'_1 \quad + c_2 f'_2 \quad + \cdots\cdots + c_n f'_n \quad \equiv 0, \\
c_1 f''_1 \quad + c_2 f''_2 \quad + \cdots\cdots + c_n f''_n \quad \equiv 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
c_1 f_1^{[n-1]} + c_2 f_2^{[n-1]} + \cdots\cdots + c_n f_n^{[n-1]} \equiv 0.
\end{array}$$

These relations may be regarded, for each particular value of x, as a system of n homogeneous linear equations satisfied by the n c's. Accordingly the determinant of the coefficients of the c's must be zero; and since this is true for all the values of x with which we have to deal, we have the identity:

$$\begin{vmatrix}
f_1 & f_2 & \cdots\cdots f_n \\
f'_1 & f'_2 & \cdots\cdots f'_n \\
\cdot & \cdot & \cdots\cdots \cdot \\
\cdot & \cdot & \cdots\cdots \cdot \\
f_1^{[n-1]} & f_2^{[n-1]} & \cdots\cdots f_n^{[n-1]}
\end{vmatrix} \equiv 0.$$

This determinant is known as the *Wronskian* of the n functions $f_1, f_2, \ldots\ldots f_n$.

We see therefore that the identical vanishing of the Wronskian is a necessary condition for linear dependence.

We will now consider under what circumstances it is also a sufficient condition. We begin with the case of two functions f_1, f_2 whose Wronskian vanishes identically:

$$f_1 f_2' - f_2 f_1' \equiv 0.$$

We will suppose at first that one of these functions, say f_1, does not vanish at any point of the region or interval in question. Then we can divide the above identity by f_1^2, after which it can be written:

$$\frac{d}{dx}\left(\frac{f_2}{f_1}\right) \equiv 0.$$

Therefore:
$$f_2 \equiv c f_1.$$

In the case of two functions, the identical vanishing of the Wronskian is a sufficient condition for linear dependence provided one of the two functions does not vanish at any point of the region in question.

This last restriction can be at once removed in the case that f_1 and f_2 are analytic functions of a complex variable. For if f_1 vanishes in the region in question, it either vanishes identically there, in which case the functions are necessarily linearly dependent, or it vanishes only at isolated points. In this latter case let us consider the region obtained from the original region, by omitting all the points at which f_1 vanishes. Throughout this new region we have, as has just been proved:

$$f_2 \equiv c f_1.$$

This relation holds throughout the neighborhood of each of the excluded points and therefore, on account of the continuity of f_1 and f_2, at these points, too; therefore:

The identical vanishing of the Wronskian of two analytic functions is a sufficient condition for linear dependence.

This is not true if we have to deal with non-analytic functions of a real variable, as the example

$$f_1 = x^2, \quad f_2 = x\,|\,x\,|$$

shows.* We will return to this case later, but proceed first to the general theorem concerning analytic functions:

The identical vanishing of the Wronskian of n analytic functions is a necessary and sufficient condition for their linear dependence.

We are concerned here merely with the proof that it is a sufficient condition. Let us assume that the theorem has been proved for the case of $n-1$ functions. If from this assumption we can deduce its proof in the case of n functions we shall have proved the theorem by the method of mathematical induction, since we know that it holds for two functions.

We start from the assumption that the Wronskian W of the n functions is identically zero. Let us denote the first minors of W which correspond to the elements of the last row by $W_1, W_2, \ldots\ldots W_n$. It should be noticed that W_i is the Wronskian of the $n-1$ functions obtained by omitting f_i. If then $W_i \equiv 0$ these $n-1$ functions are linearly dependent, and therefore (by theorem II, § 1) the n functions are linearly dependent.

Let us then assume that none of the minors W_i is identically zero. Now we have, by elementary theorems concerning determinants, the following identities:

$$\begin{array}{lllll}
W_1 f_1 & + W_2 f_2 & + \cdots + & W_n f_n & \equiv 0, \\
W_1 f_1' & + W_2 f_2' & + \cdots + & W_n f_n' & \equiv 0, \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\
W_1 f_1^{[n-1]} & + W_2 f_2^{[n-1]} & + \cdots + & W_n f_n^{[n-1]} & \equiv 0.
\end{array}$$

Differentiating each of the first $n-1$ of these identities and subtracting the next following one we get the following new set of identities:

$$\begin{array}{lllll}
W_1' f_1 & + W_2' f_2 & + \cdots + & W_n' f_n & \equiv 0, \\
W_1' f_1' & + W_2' f_2' & + \cdots + & W_n' f_n' & \equiv 0, \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\
W_1' f_1^{[n-2]} & + W_2' f_2^{[n-2]} & + \cdots + & W_n' f_n^{[n-1]} & \equiv 0.
\end{array}$$

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* $|x|$ means the absolute or numerical value of x. This example is due to Peano, *Mathesis* vol. 9, p. 110 (1889). Even if we require the existence (and therefore continuity) of derivatives of all orders at every point of the interval the theorem does not hold. *Cf. Bull. Amer. Math. Soc.*, ser. 2, vol. 7, p. 120.

Let us multiply the ith of these identities ($i = 1, 2, \ldots n-1$) by the first minor of W_n corresponding to $f_1^{[i-1]}$ and add the identities thus obtained together. We get:

$$W_1' W_n - W_1 W_n' \equiv 0.$$

Now, the f's being analytic, the W's will be analytic, and therefore we can infer from this last written identity that W_1 and W_n are linearly dependent. In precisely the same way we see that any two of the functions W_i ($i = 1, 2, \ldots n$) are linearly dependent. Since by hypothesis none of these functions are identically zero we may write:

$$W_i \equiv c_i W_n \qquad (i = 1, 2, \ldots\ldots n-1).$$

The identity

$$W_1 f_1 + W_2 f_2 + \cdots + W_n f_n \equiv 0$$

can therefore be written:

$$W_n (c_1 f_1 + c_2 f_2 + \cdots + c_{n-1} f_{n-1} + f_n) \equiv 0.$$

Since W_n is not identically zero it can vanish only at isolated points. At all other points the other factor must vanish, and therefore, owing to the continuity of the f's, it must vanish at these points, too:

$$c_1 f_1 + c_2 f_2 + \cdots\cdot + c_{n-1} f_{n-1} + f_n \equiv 0.$$

The linear dependence of the f's is thus established and our theorem is proved.

Turning now to the case of non-analytic functions of a real variable we will first prove the following preliminary theorem for the case of two functions:

The identical vanishing of the Wronskian of two functions of a real variable, each of which has a finite first derivative at every point of a certain interval, is a sufficient condition for linear dependence, provided that one of the functions and its first derivative do not vanish together at any point of the interval in question.

The interval with which we have here to deal may be infinite in one or both directions, and if limited the limiting point or points may or may not belong to the interval. Our theorem however will obviously be established if we can prove it for every finite subinterval each of whose extremities is included in the subinterval. Let ab be such a subinterval, and suppose that f_1 and f_1' do not vanish together at any point of ab, There cannot be more than a finite number of points in the interval ab where f_1 vanishes, for if there were they would have at least one limiting point x' in this interval. At the point $x' f_1$, being continuous (since it has a derivative), would vanish, and $f_1'(x') = \lim f(x' + \Delta x)/\Delta x$ would also vanish. This however is contrary to hypothesis.

The interval ab consists therefore of a finite number of pieces within each of which f_1 does not vanish. Within each of these pieces we therefore have a relation of the form:

$$f_2 \equiv c f_1 ,$$

and owing to the continuity of f_1 and f_2 this relation must also hold at the extremities of the piece in question. It remains therefore merely to prove that the constant c is the same for the different pieces; and this will be proved if we can show that it is the same for any two adjacent pieces.

Suppose then that x_0 is any point where $f_1 = 0$, and that in the piece to the right of x_0

$$f_2 \equiv c_1 f_1,$$

in the piece to the left

$$f_2 \equiv c_2 f_1.$$

Since f_1 and f_2 each has a derivative at x_0, these derivatives can be found either by differentiating to the right or to the left. We thus get:

$$f_2'(x_0) = c_1 f_1'(x_0),$$
$$f_2'(x_0) = c_2 f_1'(x_0),$$

and by subtraction:

$$(c_1 - c_2) f_1'(x_0) = 0.$$

But since f_1 vanishes at x_0, f_1' cannot vanish there, therefore: $c_1 = c_2$ and our theorem is proved.

The general theorem, of which the one just proved is a special case, is the following:

The identical vanishing of the Wronskian of n functions of a real variable, each of which has at every point of a certain interval finite derivatives of the first n $-$ 1 *orders, is a sufficient condition for linear dependence, provided that n* $-$ 1 *of the functions can be so selected that their Wronskian and its first derivative do not vanish together at any point of the interval.**

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*A theorem different from this is proved by Peano: *Rend. d. Accad. d. Lincei*, ser. 5, vol. 6, 1° sem., p. 413.

As the proof of this theorem is almost identical with the proof given above in the case of analytic functions we will not give it here in detail. It will be convenient in adapting the above proof to the present case to assume that the Wronskian which does not vanish together with its first derivative is the one which is there denoted by W_n. Since this function cannot vanish identically the step in the above proof involving mathematical induction will not here come in. On the other hand the assumption which we made before that the other Wronskians W_i do not vanish identically, and which we do not now make, was not in any way made use of in the course of the proof.

5. Applications. We will indicate, in the present paragraph, a number of applications of the subject of linear dependence to various questions, leaving the reader to work out proofs for himself.

(1). In a system of m homogeneous linear equations, between n unknowns, one at least of the k-rowed determinants of the matrix of the coefficients does not vanish, while all $(k + 1)$-rowed determinants (if any) are zero. Prove that the equations have $n - k$ linearly independent solutions, upon which every other solution is linearly dependent.

(2). DEFINITION: If we have a family of functions, of any number of variables, which can be expressed in the form:

$$k_1 f_1 + k_2 f_2 + \cdot \cdot \cdot \cdot + k_n f_n,$$

the constants k being allowed to take on all values, the family is called a *linear family* and the functions $f_1, f_2, \ldots f_n$, if they are linearly independent, are called a *basis* of the family.

Prove that if a linear family has a basis consisting of n functions:

(a) any set of $n + 1$ functions of the family are linearly dependent.

(b) no set of less than n functions can form a basis.

(c) every set of n linearly independent functions of the family forms a basis.

Obtain the most general basis from the given one.

(3) DEFINITION: A spherical harmonic is a homogeneous polynomial in the three independent variables x, y, z which satisfies Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Prove that the spherical harmonics of degree n form a linear family with a basis consisting of $2n + 1$ functions.

(4). The fundamental existence theorem for linear differential equations, when the independent variable is supposed real, is:

If throughout a certain interval $p_1, p_2, \dots \dots p_n$ are continuous functions of the real variable x, the equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdot \cdot \cdot \cdot + p_n y = 0$$

has a solution which has finite derivatives of the first n orders at every point of the interval in question, and which, at an arbitrarily chosen point of this interval has together with its derivatives of the first $n-1$ orders arbitrarily chosen values.

Assuming the truth of this theorem prove:

(a) The Wronskian of any n solutions of the equation has the value:

$$Ce^{-\int p_1 dx},$$

where C is a constant.

(b) The solutions of the equation form a linear family with a basis of n functions.

(c) The necessary and sufficient condition for the linear dependence of n solutions of the equation is the identical vanishing of the Wronskian; or, if one prefers, the vanishing of the Wronskian at some point of the interval.

(d) A solution of the equation which together with its first $n-1$ derivatives has at a point of the interval the value zero vanishes identically.

(5) Definition: the m sets of n integers each

$$x_1^{[i]}, x_2^{[i]}, \dots \dots x_n^{[i]} \qquad (i = 1, 2, \dots m)$$

are linearly dependent (mod p) if m integers $c_1, c_2, \dots \dots c_m$ not all congruent to zero (mod p) exist such that:

$$c_1 x_j' + c_2 x_j'' + \cdot \cdot \cdot + c_m x_j^{[m]} \equiv 0. \qquad (\text{mod } p).$$

Assuming the modulus p to be prime, prove the following theorem:

The above sets of integers are always linearly dependent (mod p) if $m > n$. If $m \leqq n$ the necessary and sufficient condition for linear dependence (mod p) is that all m-rowed determinants in the matrix of the sets of integers be congruent to zero (mod p).

By means of this theorem discuss the theory of simultaneous homogeneous linear congruences according to a prime modulus. (The general theory of simultaneous linear congruences has been discussed by Henry Smith in two papers *Phil. Trans.*, vol. 151, p. 293 (1861) and *Proc. London Math. Soc.*, vol. 4, p. 236 (1873); reprinted in the collected works, vol. 1, p. 367 and vol. 2, p. 67).

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